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MATHEMATICAL MODELLING OF LAMINATED PLATES ON BUCKLING

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ABSTRACT: The following assumptions were made in developing the mathematical formulations of laminated deck plates:
1. All layers behave elastically;
2. Displacements are small compared with the plate thickness;
3. Perfect bonding exists between layers;
4. The laminate is equivalent to a single anisotropic layer;
5. The plate is flat and has a constant thickness;
6. The plate buckles in a vacuum and all kinds of damping are neglected.
Unlike homogeneous plates, where the coordinates are chosen solely based on the plate shape, coordinates for laminated plates should be chosen carefully. There are two main factors for the choice of the coordinate system. The first factor is the shape of the plate. Where rectangular plates will be best represented by the choice of rectangular (i.e. Cartesian) coordinates. It will be relatively easy to represent the boundaries of such plates with coordinates. The second factor is the fiber orientation or orthotropy. If the fibers are set straight within each lamina, then rectangular orthotropy would result. It is possible to set the fibers in a radial and circular fashion, which would result in circular orthotropy. Indeed, the fibers can also be set in elliptical directions, which would result in elliptical orthotropy.

KEYWORDS: Mathematical formulation, mathematical modeling, finite element method, first order shear deformation theory, Fortran program, deck plates

1 INTRODUCTION

The choice of the coordinate system is of critical importance for laminated deck plates. This is because plates with rectangular orthotropy could be set on rectangular, triangular, circular or other boundaries. Composite materials with rectangular orthotropy are the most popular, mainly because of their ease in design and manufacturing. The equations that follow are developed for materials with rectangular orthotropy.

Figure 1.1 below shows the geometry of a plate with rectangular orthotropy drawn in the Cartesian coordinates X, Y, and Z or 1, 2, and 3. The parameters used in such a plate are: (1) the length in the X-direction, (a); (2) the length in the Y – direction (i.e. breadth), (b); and (3) the length in the Z – direction (i.e. thickness), (h).

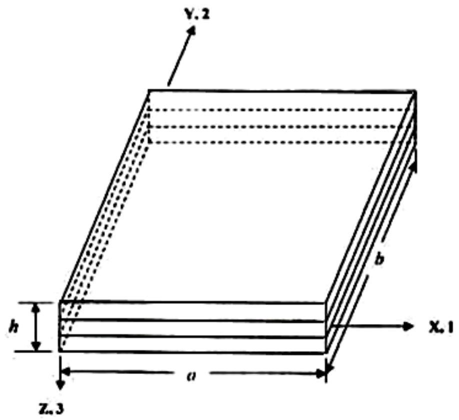


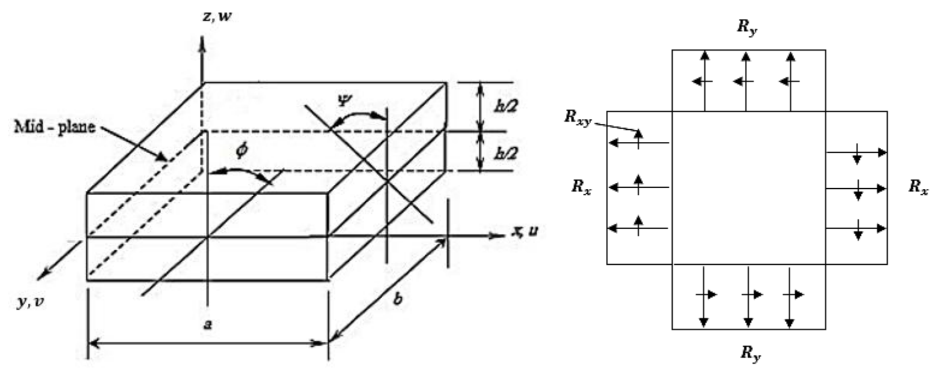
Figure 1.1 The geometry of a laminated composite plate

2 MATHEMATICAL FORMULATIONS

A first – order shear deformation theory (FSDT) is selected to formulate the problem. Consider a thin deck plate of length a, breadth b, and thickness h as shown in figure 2.1(a), subjected to in – plane loads Rx, Ry and Rxy as shown in figure 2.1(b). The in – plane displacements u (x, y, z) and v (x, y, z) can be expressed in terms of the out of plane displacement w (x, y) as shown below:
The displacements are:

$$\left.\begin{aligned} u(x,y,z) &= u_o(x,y) - z\frac{\partial w}{\partial x} \\ v(x,y,z) &= v_o(x,y) - z\frac{\partial w}{\partial x} \\ w(x,y,z) &= w_o(x,y) \end{aligned}\right\} \qquad (2.1)$$

Where u_o, v_o and w_o are mid – plane displacements in the direction of the x, y and z axes respectively; z is the perpendicular distance from mid – plane to the layer plane.



(a) (b)

Figure 2.1

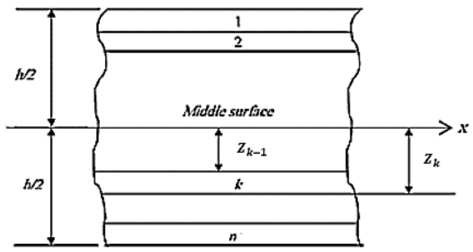


Figure 2.2 Geometry of an n-Layered laminate

The plate shown in figure 2.1 (a) is constructed of an arbitrary number of orthotropic layers bonded together as in figure 2.2 above. Refer to references [1] – [7].

The strains are:

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma &= \frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{aligned}\right\} \quad (2.2)$$

The virtual strains:

$$\left.\begin{aligned} \delta\epsilon_x &= \frac{\partial}{\partial x}\delta u_o - z\frac{\partial^2}{\partial x^2}\delta w + \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta w \\ \delta\epsilon_y &= \frac{\partial}{\partial y}\delta v_o - z\frac{\partial^2}{\partial y^2}\delta w + \frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta w \\ \delta\gamma &= \frac{\partial}{\partial x}\delta v_o + \frac{\partial}{\partial y}\delta u_o - 2z\frac{\partial^2}{\partial x\,\partial y}\delta w + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial x}\delta w\frac{\partial w}{\partial y} \end{aligned}\right\} \quad (2.3)$$

The virtual strain energy:

$$\delta U = \int_V \delta\epsilon^T \sigma dV \quad (2.4)$$

But, $\sigma = C\epsilon$

Where,

$$C = C_{ij}(i, j = 1, 2, 6)$$

$$\therefore\ \delta U = \int_V \delta\epsilon^T\, C\, \delta\epsilon\, dV \tag{2.5}$$

If we neglect the in plane displacements u_o and v_o and considering only the linear terms in the strain – displacement equations, we write:

$$\delta\epsilon = -z \begin{vmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x\,\partial y} \end{vmatrix} \delta w \tag{2.6}$$

3 NUMERICAL MODELING

The finite element is used in this analysis as a numerical method to predict the buckling loads and shape modes of buckling of laminated rectangular deck plates. In this method of analysis, four – noded type of elements are chosen. These elements are the four – noded bilinear rectangular elements of a plate. Each element has three degrees of freedom at each node. The degrees of freedom are the lateral displacement (w), and the rotations (ϕ) and (ψ) about the (X) and (Y) axes respectively.

The secondary effects of shear deformation are also considered in the present method. The shear deformation is formulated by the first – order shear deformation theory (FSDT). The finite element method is formulated by the energy method. The numerical method can be summarized in the following procedures:

The choice of the element and its shape functions.

Formulation of finite element model by the energy approach to develop both element stiffness and differential matrices.

Employment of the principles of non – dimensionality to convert the element matrices to their non – dimensionalized forms.

Assembly of both element stiffness and differential matrices to obtain the corresponding global matrices.

Introduction of boundary conditions as required for the plate edges.

Suitable software can be used to solve the problem. (here FORTRAN program was used).

For an n noded element, and 3 degrees of freedom at each node.

Now express w in terms of the shape functions N (give in Appendix (B)) and noded displacements a^e, equation (2.6) can be written as:

$$\delta\epsilon = -zB\delta a^e \tag{3.1}$$

Where,

$$B^T = \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} & \frac{\partial^2 N_i}{\partial y^2} & z\frac{\partial^2 N_i}{\partial x\,\partial y} \end{bmatrix}$$

and

$$a^e = [w_i] \quad i = 1, n$$

The stress – strain relation is:

$$\sigma = C\,\epsilon$$

Where c are the material properties which could be written as follows:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}$$

Where C_{ij} are given in Appendix (A).

$$\delta U = \int_V (B\delta a^e)^T (Cz^2) B a^e dV$$

Where V denotes volume.

$$\delta U = \delta a^{eT} \int_V B^T D B a^e dx\,dy = \delta a^{eT} K^e a^e \qquad (3.2)$$

Where $D_{ij} = \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} C_{ij} Z^2\, dZ$ is the bending stiffness, and K^e is the element stiffness matrix which could be written as:

$$K^e = \int B^T D B\, dxdy \qquad (3.3)$$

The virtual work done by external forces can be expressed as follows: Refer to Fig. (3.4).

Denoting the nonlinear part of strain by $\delta\epsilon'$

$$\delta W = \iint \delta\epsilon'^T \sigma' dV = \int \delta\epsilon'^T \overline{N}\, dxdy \qquad (3.4)$$

Where

$$N^T = [N_x N_y N_{xy}] = [\sigma_x \sigma_y\, \tau]\, dZ$$

$$\delta\epsilon' = \begin{bmatrix} \delta\epsilon_x \\ \delta\epsilon_y \\ \delta\gamma \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 \\ 0 & \frac{\partial}{\partial y}\delta w \\ \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \qquad (3.5)$$

Hence

$$\delta W = \iint \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}^T \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 & \frac{\partial}{\partial y}\delta w \\ 0 & \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} dx\,dy \qquad (3.6)$$

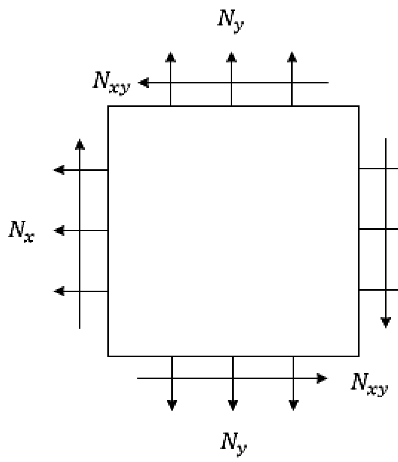


Figure 3.1 External forces acting on an element

This can be written as:

$$\delta W = \iint \begin{bmatrix} \frac{\partial}{\partial x}\delta w \\ \frac{\partial}{\partial y}\delta w \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} dx\, dy \tag{3.7}$$

Now $w = N_i a_i^e$

$$\delta W = \delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \tag{3.8}$$

Substitute $P_x = -N_x, P_y = -N_y, P_{xy} = -N_{xy}$

$$\delta W = -\delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \tag{3.9}$$

Therefore, equation (3.15) could be written in the following form:

$$\delta W = -\delta a^{eT} K^D a^e \tag{3.10}$$

Where,

$$K^D = \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} dx\, dy$$

K^D is the differential stiffness matrix known also as geometric stiffness matrix, initial stress matrix, and initial load matrix.

The total energy:

$$\delta U + \delta W = 0 \tag{3.11}$$

Since δa^e is an arbitrary displacement which is not zero, then

$$K^e a^e - K^D a^e = 0 \tag{3.12}$$

Now let us compute the elements of the stiffness and the differential matrices.

$$K^e = \iint B^T D B \, dx\, dy$$

$$K^e = \iint \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x\, \partial y} \end{bmatrix}^T \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x\, \partial y} \end{bmatrix} dx\, dy$$

The elements of the stiffness matrix can be expressed as follows:

$$K_{ij}^e = \iint \Big[D_{11} \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2} + D_{12} \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2} + 2D_{16} \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial x^2} + D_{12} \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}$$

$$D_{22} \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2} + 2D_{26} \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial y^2} + 2D_{16} \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x\, \partial y} + 2D_{26} \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x\, \partial y}$$

$$+4D_{66} \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial x\, \partial y} \Big] dx\, dy \tag{3.13}$$

The elements of the differential stiffness matrix can be expressed as follows;

$$K_{ij}^D = \iint \left[P_x \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + P_{xy}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} \right) + P_y \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right] dxdy \tag{3.14}$$

The integrals in equations (3.13) and (3.14) are given in Appendix (C).
The shape functions for a 4 – noded element is shown below in figure 3.2.

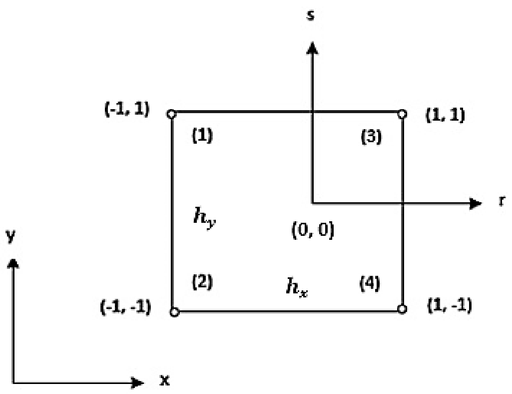


Figure 3.2 A four noded element with local and global co–ordinates

The shape functions for the 4-noded element expressed in global co-ordinates (x, y) are as follows:

$$w = N_1 w_1 + N_2 \phi_1 + N_3 \psi_1 + N_4 w_2 + N_5 \phi_2 + N_6 \psi_2$$
$$+ N_7 w_3 + N_8 \phi_3 + N_9 \psi_3 + N_{10} w_4 + N_{11} \phi_4 + N_{12} \psi_4$$

Where,

$$\phi = \frac{\partial w}{\partial x}, \qquad \psi = \frac{\partial w}{\partial y}$$

The shape functions in local co – ordinates are as follows:

$$N_i = a_{i1} + a_{i2} r + a_{i3} s + a_{i4} r^2 + a_{i5} rs + a_{i6} s^2 + a_{i7} r^3 + a_{i8} r^2 s + a_{i9} rs^2$$
$$+ a_{i10} s^3 + a_{i11} r^3 s + a_{i12} rs^3$$
$$N_j = a_{j1} + a_{j2} r + a_{j3} s + a_{j4} r^2 + a_{j5} rs + a_{j6} s^2 + a_{j7} r^3 + a_{j8} r^2 s + a_{j9} rs^2$$
$$+ a_{j10} s^3 + a_{j11} r^3 s + a_{j12} rs^3$$

The values of the coefficients a_{ij} are given in the table in Appendix (B).

$$q_1 = \iint \frac{\partial^2 N_i}{\partial r^2} \frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i4}a_{j4} + 3a_{i7}a_{j7} + \frac{1}{3}a_{i8}a_{j8} + a_{i11}a_{j11}\right]$$

$$q_2 = \iint \frac{\partial^2 N_i}{\partial s^2} \frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i6}a_{j6} + \frac{1}{3}a_{i9}a_{j9} + 3a_{i10}a_{j10} + a_{i12}a_{j12}\right]$$

$$q_3 = \iint \frac{\partial^2 N_i}{\partial r^2} \frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i4}a_{j6} + a_{i7}a_{j9} + a_{i8}a_{j10} + a_{i11}a_{j12}\right]$$

$$q_4 = \iint \frac{\partial^2 N_i}{\partial s^2} \frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i6}a_{j4} + a_{i9}a_{j7} + a_{i10}a_{j8} + a_{i12}a_{j11}\right]$$

$$q_5 = \iint \frac{\partial^2 N_i}{\partial r^2} \frac{\partial^2 N_j}{\partial r\, \partial s}\, dr\, ds = 8\left[a_{i4}a_{j5} + a_{i4}a_{j11} + 2a_{i7}a_{j8} + a_{i4}a_{j12}\right.$$
$$\left. + \frac{2}{3}a_{i4}a_{j5}\right]$$

$$q_6 = \iint \frac{\partial^2 N_i}{\partial r\, \partial s} \frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 8\left[a_{i5}a_{j4} + 2a_{i8}a_{j7} + a_{i11}a_{j4} + \frac{2}{3}a_{i9}a_{j8}\right.$$
$$\left. + a_{i12}a_{j4}\right]$$

$$q_7 = \iint \frac{\partial^2 N_i}{\partial s^2} \frac{\partial^2 N_j}{\partial r\, \partial s}\, dr\, ds = 8\left[a_{i6}a_{j5} + a_{i6}a_{j11} + \frac{2}{3}a_{i9}a_{j8}\right]$$

$$q_8 = \iint \frac{\partial^2 N_i}{\partial r\, \partial s} \frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 8\left[a_{i5}a_{j6} + \frac{2}{3}a_{i8}a_{j9} + a_{i11}a_{j6}\right]$$

$$q_9 = \iint \frac{\partial^2 N_i}{\partial r\, \partial s} \frac{\partial^2 N_j}{\partial r\, \partial s}\, dr\, ds = 4\left[a_{i5}a_{j5} + a_{i5}a_{j11} + \frac{4}{3}a_{i8}a_{j8} + a_{i5}a_{j12}\right.$$

$$+\frac{4}{3}a_{i9}a_{j9}+a_{i11}a_{j12}+a_{i12}a_{j11}+\frac{9}{5}a_{i12}a_{j12}\Big]$$

$$q_{10}=\iint\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\,dr\,ds=4\Big[a_{i2}a_{j2}+\frac{1}{3}(3a_{i2}a_{j7}+4a_{i4}a_{j4}+3a_{i7}a_{j2}$$

$$+a_{i7}a_{j9}+a_{i5}a_{j5}+a_{i5}a_{j5}+a_{i9}a_{j2}+a_{i5}a_{j11}+a_{i7}a_{j9}+\frac{4}{3}a_{i8}a_{j8}+a_{i9}a_{j7}$$

$$a_{i11}a_{j5})+\frac{1}{5}(a_{i5}a_{j12}+a_{i9}a_{j9}+a_{i12}a_{j5}+9a_{i7}a_{j7}+3a_{i11}a_{j11}+a_{i11}a_{j12}$$

$$+a_{i12}a_{j11})+\frac{1}{7}a_{i12}a_{j12}\Big]$$

$$q_{11}=\iint\frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\,dr\,ds=4\Big[a_{i3}a_{j3}+\frac{1}{3}(a_{i3}a_{j8}+a_{i5}a_{j5}+a_{i8}a_{j3}+3a_{i3}a_{j10}$$

$$+4a_{i6}a_{j6}+3a_{i10}a_{j3}+a_{i5}a_{j12}+a_{i8}a_{j10}+\frac{4}{3}a_{i9}a_{j9}+a_{i10}a_{j8}+a_{i12}a_{j5})$$

$$+\frac{1}{5}(a_{i5}a_{j11}+a_{i8}a_{j8}+a_{i11}a_{j5}+9a_{i10}a_{j10}+a_{i11}a_{j12}+a_{i12}a_{j11}+3a_{i2}a_{j12})$$

$$+\frac{1}{7}a_{i11}a_{j11}\Big]$$

$$q_{12}=\iint\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\,dr\,ds=4\Big[a_{i2}a_{j3}+\frac{1}{3}(3a_{i2}a_{j8}+2a_{i4}a_{j5}+3a_{i7}a_{j8}$$

$$+3a_{i2}a_{j10}+2a_{i5}a_{j6}+a_{i9}a_{j3}+2a_{i4}a_{j12}+3a_{i7}a_{j10}+\frac{4}{3}a_{i8}a_{j9}+\frac{1}{3}a_{i9}a_{j8}$$

$$+2a_{i11}a_{j6})\Big]$$

$$q_{13}=\iint\frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\,dr\,ds=4\Big[a_{i3}a_{j2}+\frac{1}{3}(3a_{i3}a_{j7}+2a_{i5}a_{j4}+a_{i8}a_{j2}$$

$$+a_{i3}a_{j9}+2a_{i6}a_{j5}+3a_{i10}a_{j2}+2a_{i6}a_{j11}+\frac{1}{3}a_{i8}a_{j9}+\frac{4}{3}a_{i9}a_{j8}+3a_{i10}a_{j7}$$

$$+2a_{i12}a_{j4})+\frac{1}{5}(2a_{i6}a_{j12}+3a_{i10}a_{j9}+3a_{i8}a_{j7}+2a_{i11}a_{j4})\Big]$$

The values of the integrals are converted from local co-ordinates (r, s) to global co-ordinates as follows:

$$r_1=\iint\frac{\partial^2 N_i}{\partial x^2}\frac{\partial N_j}{\partial x^2}\,dx\,dy=\left(\frac{4h_y}{h_x^3}\right)q_1=\frac{4n^3b}{ma^3}q_1$$

$$r_2=\iint\frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy=\left(\frac{4h_x}{h_y^3}\right)q_2=\frac{4am^3}{nb^3}q_2$$

$$r_3=\iint\frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy=\left(\frac{4}{h_yh_x}\right)q_3=\frac{4mn}{ab}q_3$$

$$r_4 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \left(\frac{4}{h_y h_x}\right) q_4 = \frac{4mn}{ab} q_4$$

$$r_5 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x\, \partial y}\, dx\, dy = \left(\frac{4}{h_x^2}\right) q_5 = \frac{4n^2}{a^2} q_5$$

$$r_6 = \iint \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \left(\frac{4}{h_x^2}\right) q_6 = \frac{4n^2}{a^2} q_6$$

$$r_7 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x\, \partial y}\, dx\, dy = \left(\frac{4}{h_y^2}\right) q_7 = \frac{4m^2}{a^2} q_7$$

$$r_8 = \iint \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \left(\frac{4}{h_y^2}\right) q_8 = \frac{4m^2}{b^2} q_8$$

$$r_9 = \iint \frac{\partial^2 N_i}{\partial x\, \partial y}\frac{\partial^2 N_j}{\partial x\, \partial y}\, dx\, dy = \left(\frac{4}{h_y h_x}\right) q_9 = \frac{4mn}{ab} q_9$$

$$r_{10} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx\, dy = \left(\frac{h_y}{h_x}\right) q_{10} = \frac{bn}{am} q_{10}$$

$$r_{11} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\, dx\, dy = \left(\frac{h_x}{h_y}\right) q_{11} = \frac{am}{bn} q_{11}$$

$$r_{12} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\, dx\, dy = q_{12}$$

$$r_{13} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\, dx\, dy = q_{13}$$

In the previous equations $h_x = \frac{a}{n}$ and $h_y = \frac{b}{m}$, where a and b are the lengths of the plate along the x – and y – axis respectively, n and m are the number of elements in the x – and y – directions respectively. The elements of the stiffness matrix and the differential matrix can be written as follows:

$$K_{ij} = D_{11}r_1 + D_{12}r_4 + 2D_{16}r_3 + D_{12}r^3 + D_{22}r_2 + 2D_{66}r_8 + 2D_{16}r_5$$
$$+2D_{26}r_7 + 4D_{66}r_9$$

$$K_{ij}^D = P_x r_{10} + P_{xy}(r_{12} + r_{13}) + P_y r_{11}$$

or in the non – dimensional form

$$K_{ij} = \frac{4n^3}{m}\left(\frac{b}{a}\right) D'_{11}q_1 + 4mn\left(\frac{a}{b}\right) D'_{12}q_4 + 4n^2 D'_{16}q_6 + 4mn\left(\frac{a}{b}\right) D'_{12}q_3$$
$$+\frac{4m^3}{n}\left(\frac{a}{b}\right) D'_{22}q_2 + 4m^2\left(\frac{a}{b}\right)^2 D'_{26}q_8 + 4n^2 D'_{16}q_5 + 4m^2\left(\frac{a}{b}\right)^2 D'_{26}q_7$$
$$+4mn\left(\frac{a}{b}\right) D'_{66}q_9$$

$$K_{ij}^{D} = P'_x \frac{n}{m}\left(\frac{b}{a}\right) q_{10} + P'_{xy}(q_{12} + q_{13}) + P'_y \frac{m}{n}\left(\frac{a}{b}\right) q_v$$

Where

$$D'_{ij} = \left(\frac{1}{E_2 h^3}\right) D_{ij}\,, \quad P'_i = \left(\frac{a}{E_2 h^3}\right) P_i$$

The transformed stiffness are as follows:

$$C_{11} = C'_{11}c^4 + 2c^2s^2(C'_{11} + 2C'_{66}) + C'_{22}s^4$$
$$C_{12} = c^2s^2(C'_{11} + C'_{22} + 4C'_{66}) + C'_{12}(c^4 + s^4)$$
$$C_{16} = cs[C'_{11}c^4 + C'_{22}s^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{22} = C'_{11}s^4 + 2c^2s^2(C'_{12} + 2C'_{66}) + C'_{22}c^4$$
$$C_{26} = cs[C'_{11}s^2 + C'_{22}c^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{66} = (C'_{11} + C'_{22} + 2C'_{12})c^2s^2 + C'_{66}(c^2 - s^2)^2$$

Where

$$C'_{11} = \frac{E_1}{1 - v_{12}v_{21}}$$
$$C'_{12} = \frac{v_{21}E_1}{1 - v_{12}v_{21}} = \frac{v_{12}E_1}{1 - v_{12}v_{21}}$$
$$C'_{22} = \frac{E_2}{1 - v_{12}v_{21}}$$
$$C'_{44} = G_{23}\,, \qquad C'_{55} = G_{13} \text{and} C'_{66} = G_{12}$$

E_1 and E_2 are the elastic moduli in the direction of the fiber and the transverse directions respectively, v is the Poisson's ratio. G_{12}, G_{13}, and G_{23} are the shear moduli in the $x - y$ plane, $y - z$ plane, and $x - z$ plane respectively, and the subscripts 1 and 2 refer to the direction of fiber and the transverse direction respectively.

4 CONCLUSIONS

Finite element method (FEM) was used so as to predict the buckling loads and shape modes of laminated rectangular deck plates. A suitable element type is chosen and its shape functions are determined. Energy approach is used to formulate the finite element model and develop both element stiffness and differential matrices. These matrices are assembled to give the corresponding global matrices, the required boundary conditions are introduced and a suitable software (i.e. Fortran) is used to solve the problem.

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APPENDICES

Appendix (A)

The transformed material properties are:

$$C_{11} = C'_{11}\cos^4\theta + C'_{22}\sin^4\theta + 2(C'_{12} + 2C'_{66})\sin^2\theta\cos^2\theta$$

$$C_{12} = \left(C'_{11} + C'_{22} - 4C'_{66}\right)\sin^2\theta\cos^2\theta + C^{1}_{12}(\cos^4\theta + \sin^4\theta)$$

$$C_{22} = C'_{11}\sin^4\theta + C'_{22}\cos^4\theta + 2(C'_{12} + 2C'_{66})\sin^2\theta\cos^2\theta$$

$$C_{16} = \left(C'_{11} - C'_{12} - 2C'_{66}\right)\cos^3\theta\sin\theta - \left(C'_{22} - C'_{12} - 2C'_{66}\right)\sin^3\theta\cos\theta$$

$$C_{26} = \left(C'_{11} - C'_{12} - 2C'_{66}\right)\cos\theta\sin^3\theta - \left(C'_{22} - C'_{12} - 2C'_{66}\right)\sin\theta\cos^3\theta$$

$$C_{66} = \left(C'_{11} + C'_{22} - 2C'_{12} - 2C'_{66}\right)\sin^2\theta\cos^2\theta + C'_{66}(\sin^4\theta + \cos^4\theta)$$

$$\text{where} C'_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, C'_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, C'_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, C'_{16} = G_{12}$$

Appendix (B)

$a_{i,j}/8$

i / N_i	$i,1$	$i,2$	$i,3$	$i,4$	$i,5$	$i,6$	$i,7$	$i,8$	$i,9$	$i,10$	$i,11$	$i,12$
N_1	2	-3	3	0	-4	0	1	0	0	-1	1	1
N_2	1	-1	1	-1	-1	0	1	-1	0	0	1	0
N_3	-1	1	-1	0	1	1	0	0	-1	1	0	-1
N_4	2	-3	-3	0	4	0	1	0	0	1	-1	-1
N_5	1	-1	-1	-1	1	0	1	1	0	0	-1	0
N_6	1	-1	-1	0	1	-1	0	0	1	1	0	-1
N_7	2	3	3	0	4	0	-1	0	0	-1	-1	-1
N_8	-1	-1	-1	1	-1	0	1	1	0	0	1	0
N_9	-1	-1	-1	0	-1	1	0	0	1	1	0	1
N_{10}	2	3	-3	0	-4	0	-1	0	0	1	1	1
N_{11}	-1	-1	1	1	1	0	1	-1	0	0	-1	0
N_{12}	1	1	-1	0	-1	-1	0	0	-1	1	0	1

Appendix (C)

The integrals in equations (13) and (14) are given in nondimensional form as follows (limits of integration $r, s = -1$ to1):

$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2}\,dx\,dy = \frac{4h_y}{{h_x}^3}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r^2}\,dr\,ds$$

$$= \frac{4n^3}{mR}\left(16a_{i,4}a_{j,4} + 48a_{i,7}a_{j,7} + 16a_{i,8}a_{j,8}/3 + 16a_{i,11}a_{j,11}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy = \frac{4h_x}{{h_y}^3}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial s^2}\,dr\,ds$$

$$= \frac{4m^3R^3}{n}\left(16a_{i,6}a_{j,6} + 16a_{i,9}a_{j,9}/3 + 48a_{i,10}a_{j,10} + 16a_{i,12}a_{j,12}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy = \frac{4}{h_y h_x}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial s^2}\,dr\,ds$$

$$= 4mnR\left(16a_{i,4}a_{j,6} + 16a_{i,7}a_{j,9} + 16a_{i,8}a_{j,10} + 16a_{i,11}a_{j,12}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\,dx\,dy = \frac{4}{h_y h_x}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r^2}\,dr\,ds$$

$$= 4mnR\left(16a_{i,6}a_{j,4} + 16a_{i,9}a_{j,7} + 16a_{i,10}a_{j,8} + 16a_{i,12}a_{j,11}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial x\,\partial y}\frac{\partial^2 N_j}{\partial x\,\partial y}\,dx\,dy = \frac{4}{h_y h_x}\iint \frac{\partial^2 N_i}{\partial r\,\partial s}\frac{\partial^2 N_j}{\partial r\,\partial s}\,dr\,ds =$$

$$4mnR\big[4a_{i,5}a_{j,5} + 4(3a_{i,5}a_{j,11} + 4a_{i,8}a_{j,8})/3$$
$$+4(3a_{i,5}a_{j,12} + 4a_{i,9}a_{j,9})/3 + 4(a_{i,11}a_{j,12} + a_{i,12}a_{j,11}) + 36a_{i,12}a_{j,12}/5\big]$$
$$\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dx\,dy = \frac{h_y}{h_x}\iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\,dr\,ds$$
$$= \frac{n}{mR}\big[4a_{i,2}a_{j,2} + 4(3a_{i,2}a_{j,7} + 4a_{i,4}a_{j,4} + 3a_{i,7}a_{j,2})/3$$
$$+4(a_{i,2}a_{j,9} + a_{i,5}a_{j,5} + a_{i,9}a_{j,2})/3 + 4(3a_{i,5}a_{j,11} + 3a_{i,7}a_{j,9} + 4a_{i,8}a_{j,8}$$
$$+3a_{i,9}a_{j,7} + 3a_{i,11}a_{j,5})/9 + 4(a_{i,5}a_{j,12} + a_{i,9}a_{j,9} + a_{i,12}a_{j,5})/5$$
$$+36a_{i,7}a_{j,7}/5 + 12a_{i,11}a_{j,11}/5 + 4(a_{i,11}a_{j,12} + a_{i,12}a_{j,11})/5 + 4a_{i,12}a_{j,12}/7\big]$$
$$\iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,dx\,dy = \frac{h_x}{h_y}\iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\,dr\,ds$$
$$= \frac{mR}{n}\big[4a_{i,3}a_{j,3} + 4(a_{i,3}a_{j,8} + a_{i,5}a_{j,5} + a_{i,8}a_{j,3})/3$$
$$+4(3a_{i,3}a_{j,10} + 4a_{i,6}a_{j,6} + 3a_{i,10}a_{j,3})/3 + 4(3a_{i,5}a_{j,11} + a_{i,8}a_{j,8} + a_{i,11}a_{j,5})/5$$
$$+4(3a_{i,5}a_{j,12} + 3a_{i,8}a_{j,10} + 4a_{i,9}a_{j,9} + 3a_{i,10}a_{j,8} + 3a_{i,12}a_{j,5})/9$$
$$+36a_{i,10}a_{j,10}/5 + 4(a_{i,11}a_{j,12} + a_{i,12}a_{j,11})/5 + 12a_{i,12}a_{j,12}/5 + 4a_{i,11}a_{j,11}/7]$$
$$\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\,dx\,dy = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial s}\,dr\,ds$$
$$= 4a_{i,2}a_{j,3} + 4(a_{i,2}a_{j,8} + 2a_{i,4}a_{j,5} + 3a_{i,7}a_{j,8})/3 + 4(3\,a_{i,2}a_{j,10} + 2a_{i,5}a_{j,6}$$
$$+a_{i,9}a_{j,3})/3 + 4(2a_{i,4}a_{j,11} + 3a_{i,7}a_{j,8})/5 + 4(6a_{i,4}a_{j,12} + 9a_{i,7}a_{j,10}$$
$$+4a_{i,8}a_{j,9} + a_{i,9}a_{j,8} + 6a_{i,11}a_{j,6})/9 + 4(3a_{i,9}a_{j,10} + 2a_{i,12}a_{j,6})/5$$
$$\iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\,dx\,dy = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\,dr\,ds$$
$$= 4a_{i,3}a_{j,2} + 4(3a_{i,3}a_{j,7} + 2a_{i,5}a_{j,4} + a_{i,8}a_{j,2})/3 + 4(\,a_{i,3}a_{j,9} + 2a_{i,6}a_{j,5}$$
$$+3a_{i,10}a_{j,2})/3 + 4(6a_{i,6}a_{j,11} + a_{i,8}a_{j,9} + 4a_{i,9}a_{j,8} + 9a_{i,10}a_{j,7} + 6a_{i,2}a_{j,4})/9$$
$$+4(2a_{i,6}a_{j,12} + 3a_{i,10}a_{j,9})/5 + 4(3a_{i,8}a_{j,7} + 2a_{i,11}a_{j,4})/5$$
$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x\,\partial y}\,dx\,dy = \frac{4}{{h_x}^2}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r\,\partial s}\,dr\,ds$$
$$= 4n^2\big[8a_{i,4}\big(a_{j,5} + a_{j,11} + a_{j,12}\big) + 16(a_{i,7}a_{j,8} + a_{i,8}a_{j,9}/3)\big]$$
$$\iint \frac{\partial^2 N_i}{\partial x\,\partial y}\frac{\partial^2 N_j}{\partial x^2}\,dx\,dy = \frac{4}{{h_x}^2}\iint \frac{\partial^2 N_i}{\partial r\,\partial s}\frac{\partial^2 N_j}{\partial r^2}\,dr\,ds$$
$$= 4n^2\big[8a_{j,4}\big(a_{i,5} + a_{i,11} + a_{i,12}\big) + 16a_{i,8}a_{j,7} + 16a_{i,9}a_{j,8}/3\big]$$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x\,\partial y}\,dx\,dy = \frac{4}{{h_y}^2}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r\,\partial s}\,dr\,ds$$

$$= 4m^2R^2\left[8a_{i,6}\left(a_{j,5} + a_{j,11} + a_{j,12}\right) + 16a_{i,10}a_{j,9} + 16a_{i,9}a_{j,8}/3\right]$$

$$\iint \frac{\partial^2 N_i}{\partial x\,\partial y}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy = \frac{4}{{h_y}^2}\iint \frac{\partial^2 N_i}{\partial r\,\partial s}\frac{\partial^2 N_j}{\partial s^2}\,dr\,ds$$

$$= 4m^2R^2\left[8a_{j,6}\left(a_{i,5} + a_{i,11} + a_{i,12}\right) + 16a_{i,9}a_{j,10} + 16a_{i,8}a_{j,9}/3\right]$$

In the above expressions $h_x = \frac{a}{n}$, $h_y = \frac{b}{m}$ where a and b are the dimensions of the plate in the x – and y – directions respectively. n and m are the number of elements in the x – and y – directions respectively. Note that $dx = \frac{h_x}{2}dr$ and $dy = \frac{h_y}{2}ds$ where r and s are the normalized coordinates, and $R = a/b$.